Clairaut-Schwarz theorem

Given the function:

$$f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

1. Compute f'_x and f'_y for $(x,y) \neq (0,0)$
2. Prove that $f'_x(0,0) = f'_y(0,0) = 0$
3. Prove that $f''_{xy} \neq f''_{xy}(0,0)$
4. Why are the mixed partial derivatives not equal? Does Schwarz's theorem not hold?

Solutions

1. Compute by using the chain rule:

$$f'_x = \frac{(3x^2y - y^3)(x^2 + y^2) - (2x)(x^3y - xy^3)}{(x^2 + y^2)^2}$$

$$f'_y = \frac{(x^3 - 3xy^2)(x^2 + y^2) - (2y)(x^3y - xy^3)}{(x^2 + y^2)^2}$$

2. Compute by definition:

$$f'_x(0,0) = \lim_{h\to 0} \frac{\dfrac{(0+h)^3 \times 0 - (0+h) \times 0^3}{(0+h)^2 + 0^2} - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f'_y(0,0) = \lim_{h\to 0} \frac{\dfrac{0^3 \times (0+h) - 0 \times (0+h)^3}{0^2 + (0+h)^2} - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

3. Using the first derivatives, we now calculate the mixed partial derivatives by definition:

$$f''_{xy} = \lim_{h\to 0} \frac{\dfrac{(3 \times 0^2 \times (0+h) - (0+h)^3) \times (0^2 + (0+h)^2) - (2 \times 0) \times (0^3 \times (0+h) - 0 \times (0+h)^3)}{(0^2 + (0+h)^2)^2} - f'_x(0,0)}{h}$$

$$f''_{xy} = \lim_{h\to 0} \frac{\dfrac{-h^3 \times h^2 - 0}{h^4} - 0}{h} = \lim_{h\to 0} \frac{-h^5}{h^5} = -1$$

$$f''_{yx} = \lim_{h\to 0} \frac{\dfrac{((0+h)^3 - 3 \times (0+h) \times 0^2) \times ((0+h)^2 + 0^2) - (2 \times 0) \times ((0+h)^3 \times 0 - (0+h) \times 0^3)}{((0+h)^2 + 0^2)^2} - f'_y(0,0)}{h}$$

$$f''_{yx} = \lim_{h\to 0} \frac{\dfrac{h^3 \times h^2}{h^4} - 0}{h} = 1$$

4. Schwarz's theorem does not hold because the mixed partial derivatives of the function are not continuous at the point.